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Compact solvmanifolds with calibrated and cocalibrated G_2-structures

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Abstract. We give a method to obtain new solvable 7-dimensional Lie algebras endowed with closed and coclosed G_2-structures starting from 6-dimensional solvable Lie algebras with symplectic half-flat and half-flat SU(3)-structures, respectively. Provided the existence of a lattice for the corresponding Lie groups we obtain new examples of compact solvmanifolds endowed with calibrated and cocalibrated G_2-structures. As an application of this construction we also obtain a formal compact solvmanifold with first Betti number $b_1 = 1$ endowed with a calibrated G_2-structure and such that does not admit any invariant torsion-free G_2-structure.

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Introduction

A G_2-structure on a 7-dimensional manifold M consists in a reduction of its frame bundle to the exceptional Lie group G_2. Such structure can also be characterized by the existence of a global non-degenerate 3-form φ on M, which is called the fundamental form or G_2-form. As it is described in [13] the presence of such structure on a manifold induces a two-fold vector cross product P, a metric g_φ and a volume form vol, satisfying

$$g_\varphi(P(X,Y),Z) = \varphi(X,Y,Z)$$

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for all $X, Y, Z \in \mathfrak{X}(M)$.

Such structure allows to distinguish also a 4-form that can be described as the Hodge star $*$ of the fundamental form.

In [15] Fernández and Gray obtained a classification of G_2 manifolds attending to the decomposition of the covariant derivative of the fundamental form into G_2 irreducible components. They described 16 different classes of G_2 manifolds, among which we must distinguish calibrated G_2 manifolds (the fundamental form is closed, i.e. $d\varphi = 0$) and cocalibrated G_2 ones ($d * \varphi = 0$). This two classes are two of the most important ones since if the fundamental form is closed and coclosed the manifold has holonomy on G_2, see [15]. A G_2-structure φ such that is calibrated and cocalibrated at the same time is usually called parallel or torsion-free. Obtaining compact examples of these classes is not an easy task. The first example of compact calibrated G_2 manifold was given in [12] and consists on a nil-manifold (compact quotient of a nilpotent Lie group by a lattice). Since then, many other examples have been obtained but there has been a clear absence of examples with certain properties like formal ones or examples with first Betti number lower than 2. Recently in [24, Section 1.4] and [14] examples satisfying both conditions simultaneously have been obtained. In particular, in [14] the authors show that the example there described does not admit any torsion-free G_2-structure.

In this paper we focus our attention on the construction of compact manifolds endowed with closed and coclosed G_2-structures. In particular we obtain compact solvmanifolds (compact quotients of solvable Lie groups by a lattice) endowed with that structures. In order to obtain these examples we describe first how to obtain closed and coclosed G_2-structures on solvable Lie algebras.

It is well-known that the presence of a symplectic half-flat structure namely (ω, ψ_+) on a 6-dimensional Lie algebra $\mathfrak{h}$, defines a closed G_2-structure on $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$. Equivalently, if the $SU(3)$-structure (ω, ψ_+) on $\mathfrak{h}$ is half-flat, a coclosed G_2-structure can be defined on $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$. In the present work we generalize this well-known construction. This fact allows to obtain new examples of 7-dimensional Lie algebras endowed with closed and coclosed G_2-structures. Thus, provided the existence of a lattice we can construct new compact solvmanifolds endowed with special G_2-structures.

This work is structured as follows: Sect. 1 is devoted to recall some preliminars on $SU(3)$ and G_2-structures. In Sect. 2 we also recall some facts concerning minimal models and formality. Section 3 is focused on the construction of new 7-dimensional solvable Lie algebras endowed with a closed G_2-structure. In particular in Theorem 3.1 we describe how to obtain a 7-dimensional Lie algebra of the form

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R} \tag{1}$$

endowed with a closed G_2-structure from a 6-dimensional Lie algebra $\mathfrak{h}$ with a symplectic half-flat $SU(3)$-structure, where D denotes a derivation of $\mathfrak{h}$, what constitutes a generalization of some of the results obtained in [19] for almost abelian algebras. Note that almost abelian Lie algebras have also been considered by several authors with different purposes. Lauret in [23] use them in order to obtain almost abelian solvmanifolds endowed with G_2-structures solving the Laplacian flow. Also in [2] the authors consider almost abelian Lie algebras to find explicit solutions of the Laplacian coflow of G_2-structures on 7-dimensional almost abelian Lie groups.

In order to obtain many new examples of Lie algebras endowed with closed G_2-structures we consider all the 6-dimensional solvable Lie algebras endowed with a symplectic half-flat SU(3)-structure (obtained in [16]) and thus apply the previously mentioned construction. Finally in subsections 3.1 and 3.2 we show, compact G_2 calibrated manifolds, which are respectively an almost nilpotent one and a formal almost abelian example with first Betti number equal to 1 not admitting invariant torsion-free G_2-structures. As far as the author knows this latter example is the first example in these conditions in the class of compact solvmanifolds.

Section 4 is devoted to an equivalent study considering coclosed G_2-structures and half-flat SU(3)-structures. In particular, in Theorem 4.1 we describe how to obtain 7-dimensional Lie algebras endowed with a coclosed G_2-structure from 6-dimensional Lie algebras with half-flat SU(3)-structures. In subsections 4.1, and 4.2 we show, compact G_2 cocalibrated manifolds, which are respectively an almost abelian, an almost nilpotent one.

1. Prelimnars on SU(3) and G_2-structures

An $SU(n)$-structure on a Lie algebra $\mathfrak{h}$ of dimension $2n$, consists in a triple (g, J, Ψ) such that (g, J) is an almost Hermitian structure on $\mathfrak{h}$, and $\Psi = \psi_+ + i\,\psi_-$ is a complex volume $(n, 0)$-form, satisfying

$$(-1)^{n(n-1)/2}\left(\frac{i}{2}\right)^n \Psi \wedge \overline{\Psi} = \frac{1}{n!}\,\omega^n,$$

with $\overline{\Psi}$ the complex form obtained by conjugation of Ψ, and ω the Kähler form associated to (g, J). In what follows we will consider SU(3)-structures on 6-dimensional Lie algebras.

The existence of an SU(3)-structure on a Lie algebra $\mathfrak{h}$ can also be described by the presence of a pair of forms, namely, $(\omega, \psi_+) \in \Lambda^2\mathfrak{h}^* \times \Lambda^3\mathfrak{h}^*$ such that describe a metric as

$$g(X, Y)\,\omega^3 = -3\,\iota_X\omega \wedge \iota_Y(\psi_+) \wedge \psi_+,$$

with $X, Y \in \mathfrak{h}$ and ι_X denoting the contraction by X. We can also recover its compatible almost complex structure as it is described in [10]

$$-2(J^*_{\psi_+}\alpha)(X)\,\frac{\omega^3}{3!} = \alpha \wedge \iota_X\psi_+ \wedge \psi_+,$$

or, equivalently,

$$\alpha(JX) = -J^*\alpha(X),$$

for any 1-form α on $\mathfrak{h}^*$.

Also, if (g, J, Ψ) is an SU(3)-structure on a Lie algebra $\mathfrak{h}$ we may choose an orthonormal frame $\{e_1, \ldots, e_6\}$ such that the almost complex structure J is

$J^*e^1 = e^2$, $J^*e^3 = e^4$ and $J^*e^5 = e^6$ with $\{e^1, \ldots, e^6\}$ an orthonormal basis dual to $\{e_1, \ldots, e_6\}$. Therefore, the Kähler form ω and the complex volume form Ψ can be written as

$$\omega = e^{12} + e^{34} + e^{56}, \qquad \Psi = (e^1 + i\, e^2) \wedge (e^3 + i\, e^4) \wedge (e^5 + i\, e^6), \tag{2}$$

where, with the usual notation of the related literature, we write e^{ij} for the wedge product $e^i \wedge e^j$, $e^{ijk} = e^i \wedge e^j \wedge e^k$, and so on. Thus,

$$\psi_+ = e^{135} - e^{146} - e^{236} - e^{245}, \qquad \text{and} \qquad \psi_- = -e^{246} + e^{235} + e^{145} + e^{136}.$$

In [21], Gray and Hervella prove that there exist sixteen different classes of almost Hermitian structures according to the behavior of the covariant derivative of its Kähler form. Equivalently, the different classes of $\mathrm{SU}(n)$-structures can be defined in terms of the forms ω, ψ_+ and ψ_-. In particular we are interested in two classes of $\mathrm{SU}(3)$-structures which were defined respectively in [8] and [25] as follows:

- (g, J, Ψ) is a *half-flat* $\mathrm{SU}(3)$-structure iff $d\omega^2 = d\psi_+ = 0$;
- (g, J, Ψ) is a *symplectic half-flat* $\mathrm{SU}(3)$-structure iff $d\omega = d\psi_+ = 0$;

A classification of half-flat $\mathrm{SU}(3)$-structures on nilpotent Lie algebras is done in [6]. In [18] a similar work for decomposable solvable Lie algebras has been established. The existence of symplectic half-flat $\mathrm{SU}(3)$-structures on nilpotent Lie algebras is studied in [9] and the complete study of these structures on solvable Lie algebras is obtained in [16].

A G_2-structure on a 7-dimensional Lie algebra $\mathfrak{g}$ is defined by a 3-form φ (called the fundamental form) on $\mathfrak{g}$ which also induces a metric g_φ and a volume form vol satisfying

$$g_\varphi(X, Y)\, vol = \frac{1}{6}\, \iota_X\varphi \wedge \iota_Y\varphi \wedge \varphi,$$

for all $X, Y \in \mathfrak{g}$. With respect to some orthonormal basis of 1-forms $\{e^1, \ldots, e^7\}$ on $\mathfrak{g}$ the fundamental form can be written as

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}. \tag{3}$$

It can also be defined the 4-form $*\varphi$, where $*$ denotes the Hodge star operator associated to g_φ. Therefore, respect to the basis $\{e^1, \ldots, e^7\}$ of 1-forms of $\mathfrak{g}$ in which the fundamental form is described by (3) the 4-form can be described as

$$\varphi = e^{1234} + e^{1256} + e^{1234} - e^{2467} + e^{2357} + e^{1457} + e^{1367}.$$

In [15], Fernández and Gray prove that there exist sixteen different classes of G_2-structures according to the behavior of the covariant derivative of its fundamental form. In particular we will be interested in two different classes of G_2-structures which are described as follows:

- φ is an *almost parallel* or *closed* G_2-structure iff $d\varphi = 0$;
- φ is a *semiparallel* or *coclosed* G_2-structure iff $d * \varphi = 0$;

A classification of closed G_2-structures on nilpotent Lie algebras has been obtained in [7]. SU(3) and G_2-structures are closely related. In fact, if (N^6, ω, ψ_+) is a 6-dimensional manifold endowed with an SU(3)-structure then the 3-form

$$\varphi = \omega \wedge dt + \psi_+, \tag{4}$$

defines a G_2-structure on the 7-dimensional manifold $M^7 = N^6 \times S^1$ where t denotes the coordinate in S^1.

Concerning the relation between special SU(3)-structures and special G_2-structures, if the SU(3)-structure (ω, ψ_+) on N^6 is symplectic half-flat clearly the G_2-structure defined by (4) constitutes a closed G_2-structure on M^7. Equivalently, if the SU(3) manifold (N^6, ω, ψ_+) is half-flat the 3-form

$$\varphi = \omega \wedge dt - \psi_-, \tag{5}$$

is such that

$$*\varphi = \frac{1}{2}\omega \wedge \omega + \psi_+ \wedge dt,$$

and therefore defines a coclosed G_2-structure on the 7-dimensional manifold $M^7 = N^6 \times S^1$, where t is the coordinate on S^1.

2. Minimal models and formality

In this section some definitions and results about minimal models and formality are reviewed. All these facts are very well known in the literature and can be found, for example, in [11,14,17].

From now on, we work with graded algebras over the field of real numbers $\mathbb{R}$, and we denote by $|a|$ the degree of an element.

A *differential graded commutative algebra* $(\mathcal{A}, d)$ over $\mathbb{R}$ (CDGA for short) consists on pair $(\mathcal{A}, d)$, where $\mathcal{A}$ is a graded commutative algebra $\mathcal{A} = \oplus_{i\geq 0} A^i$ over $\mathbb{R}$, and $d\colon A^* \to A^{*+1}$ is a derivation of degree 1, that is, d is a linear map such that $d^2 = 0$ and, for homogeneous elements a and b,

$$d(a \cdot b) = (da) \cdot b + (-1)^{|a|} a \cdot (db).$$

Given a differential graded commutative algebra $(\mathcal{A},\ d)$, we denote its cohomology by $H^*(\mathcal{A})$. The cohomology of a differential graded algebra $H^*(\mathcal{A})$ is a CDGA with the product inherited from that on $\mathcal{A}$ and with the differential being identically zero. The CDGA $(\mathcal{A},\ d)$ is *connected* if $H^0(\mathcal{A}) = \mathbb{R}$, and $(\mathcal{A},\ d)$ is 1-*connected* if, in addition, $H^1(\mathcal{A}) = 0$.

In our context, the main examples of CDGAs are the de Rham complex $(\Omega^*(M),\ d)$ of a differentiable manifold M, where d is the exterior differential, and the de Rham cohomology algebra $(H^*(M), d = 0)$.

If $(\mathcal{A}, d_{\mathcal{A}})$ and $(\mathcal{B}, d_{\mathcal{B}})$ are CDGAs, a map

$$\nu : (\mathcal{A}, d_{\mathcal{A}}) \longrightarrow (\mathcal{B}, d_{\mathcal{B}}),$$

is called *morphism of CDGA's* if ν is a morphism of algebras such that preserves the degree and commutes with the differential.

Definition 2.1. A CDGA $(\mathcal{A}, d)$ is said to be *minimal* if

- $\mathcal{A}$ is the free algebra $\mathcal{A} = \bigwedge V$ over a graded (real) vector space $V = \oplus_k V^k$; and
- there exists a basis $\{x_i, i \in I\}$ of V, for a well-ordered index set I, such that $|x_i| \leq |x_j|$ if $i < j$, and each dx_j is expressed in terms of the preceding x_i $(i < j)$.

Definition 2.2. Let $(\mathcal{M}, d_{\mathcal{M}})$ and $(\mathcal{A}, d)$ be two CDGA's. We say that $(\mathcal{M}, d_{\mathcal{M}})$ is a *minimal model* of $(\mathcal{A}, d)$ if $(\mathcal{M}, d_{\mathcal{M}})$ is minimal, so $\mathcal{M} = \bigwedge V$, and there exists a morphism

$$\rho : (\mathcal{M}, d_{\mathcal{M}}) \longrightarrow (\mathcal{A}, d),$$

of DGAs, such that it induces an isomorphism in cohomology

$$\rho^* : H^*(\mathcal{M}) \xrightarrow{\cong} H^*(\mathcal{A}).$$

In [22], Halperin proved that any connected differential graded algebra $(\mathcal{A},\ d)$ has a minimal model unique up to isomorphism. For 1-connected differential algebras, a similar result was proved earlier by Deligne, Griffiths, Morgan and Sullivan [11].

Definition 2.3. A *minimal model* of a connected differentiable manifold M is a minimal model $(\mathcal{M},\ d_{\mathcal{M}})$ of the de Rham complex $(\Omega^*(M),\ d)$ of differential forms on M.

Definition 2.4. A minimal model $(\mathcal{M},\ d_{\mathcal{M}})$ is *formal* if there exists a morphism of differential algebras

$$\psi\colon \big(\mathcal{M},\ d_{\mathcal{M}}\big) \longrightarrow \big(H^*(\mathcal{M}), 0\big),$$

inducing the identity map on cohomology. Also a differentiable manifold M is called *formal* if its minimal model is formal.

The formality of a minimal algebra is characterized as follows.

Theorem 2.5. ([11]) *A minimal algebra $\big(\mathcal{M},\ d_{\mathcal{M}}\big)$ with $\mathcal{M} = \bigwedge V$ is formal if and only if the space V can be decomposed into a direct sum $V = C \oplus N$ with $d(C) = 0$ and d injective on N, such that every closed element in the ideal $I(N)$ in $\bigwedge V$ generated by N is exact.*

This characterization of formality can be weakened using the concept of s-formality introduced in [17].

Definition 2.6. A minimal algebra $\big(\mathcal{M},\ d_{\mathcal{M}}\big)$ with $\mathcal{M} = \bigwedge V$ is s-formal $(s > 0)$ if for each $i \leq s$ the space V^i of generators of degree i decomposes as a direct sum $V^i = C^i \oplus N^i$, where the spaces C^i and N^i satisfy the three following conditions:

(1) $d(C^i) = 0$,
(2) the differential map $d\colon N^i \longrightarrow \bigwedge V$ is injective, and

(3) any closed element in the ideal $I_s = I(\bigoplus_{i \le s} N^i)$, generated by the space $\bigoplus_{i \le s} N^i$ in the free algebra $\bigwedge(\bigoplus_{i \le s} V^i)$, is exact in $\bigwedge V$.

A differentiable manifold M is *s-formal* if its minimal model is s-formal. Clearly, if M is formal then M is s-formal, for any $s > 0$. The main result of [17] shows that formality can be guaranteed or discarded with the weaker condition of s-formality.

Theorem 2.7. ([17]) *Let M be a connected and orientable compact differentiable manifold of dimension $2n$ or $(2n-1)$. Then M is formal if and only if it is $(n-1)$-formal.*

3. Lie algebras with a calibrated G_2-structure

Consider $\mathfrak{h}$ a 6-dimensional Lie algebra, and D a derivation of $\mathfrak{h}$, thus the vector space

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}\xi$$

is a Lie algebra with the Lie bracket given by

$$[U, V] = [U, V]|_{\mathfrak{h}}, \qquad [\xi, U] = D(U),$$

for any $U, V \in \mathfrak{h}$.

Let (ω, ψ_+) be a symplectic half-flat structure on $\mathfrak{h}$. Thus, it defines an almost complex structure J, and as it is mentioned in [4] this allows to obtain a real representation of the complex matrices as

$$\rho : \mathfrak{gl}(3, \mathbb{C}) \longrightarrow \mathfrak{gl}(6, \mathbb{R}).$$

Then, if $A \in \mathfrak{gl}(3, \mathbb{C})$, $\rho(A)$ is the matrix $(B_{ij})_{i,j=1}^3$ with

$$B_{ij} = \begin{pmatrix} ReA_{ij} & ImA_{ij} \\ -ImA_{ij} & ReA_{ij} \end{pmatrix},$$

where A_{ij} is the (i, j) component of A.
In particular, let us recall that the real representation of $\mathfrak{sl}(3, \mathbb{C})$ (complex matrices without trace) is given by

$$\mathfrak{sl}(3, \mathbb{C}) = \left\{ \left(\begin{array}{cc|cc|cc} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ -a_{1,2} & a_{1,1} & -a_{1,4} & a_{1,3} & -a_{1,6} & a_{1,5} \\ \hline a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} \\ -a_{3,2} & a_{3,1} & -a_{3,4} & a_{3,3} & -a_{3,6} & a_{3,5} \\ \hline a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & -a_{1,1} - a_{3,3} & -a_{1,2} - a_{3,4} \\ -a_{5,2} & a_{5,1} & -a_{5,4} & a_{5,3} & a_{1,2} + a_{3,4} & -a_{1,1} - a_{3,3} \end{array} \right), \text{ with } a_{i,j} \in \mathbb{R} \right\}. \tag{6}$$

Theorem 3.1. *Let $\mathfrak{h}$ be a 6-dimensional Lie algebra and let $\mathfrak{g}$ be a 7-dimensional Lie algebra satisfying*

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7,$$

with D a derivation of $\mathfrak{h}$ such that $D \in \mathfrak{sl}(3, \mathbb{C})$, then the following two conditions are equivalent:

(1) *The* SU(3)*-structure on $\mathfrak{h}$ given by*

$$\omega = e^{12} + e^{34} + e^{56},$$
$$\psi_+ = e^{135} - e^{146} - e^{236} - e^{245}.$$

is symplectic half-flat.

(2) *The* G_2*-structure on $\mathfrak{g}$ given by*

$$\varphi = \omega \wedge e^7 + \psi_+,$$

is closed.

Proof. Identifying k-forms on $\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}\xi$ which annihilate ξ with k-forms on $\mathfrak{h}$, one may write any k-form $\gamma \in \Lambda^k \mathfrak{g}^*$ as

$$\gamma = \alpha \wedge \xi^\flat + \beta$$

for unique $\alpha \in \Lambda^{k-1}\mathfrak{h}^*$ and $\beta \in \Lambda^k \mathfrak{h}^*$ where $\flat$ denotes the canonical isomorphism. One can check that

$$d_\mathfrak{g}\gamma = d_\mathfrak{h}\alpha \wedge \xi^\flat + \xi^\flat \wedge D.\beta + d_\mathfrak{h}\beta \tag{7}$$

for $D.\beta$ being the natural action of $D \in \mathfrak{gl}(\mathfrak{h})$ on $\beta \in \Lambda^k \mathfrak{h}^*$.

Thus, consider the SU(3)-structure (ω, ψ_+) on $\mathfrak{h}$ such that with respect to the basis $\{e_1, \ldots, e_6\}$ has the canonical expression. Consider also, the G_2 form

$$\varphi = \omega \wedge \eta + \psi_+,$$

with η the 1-form such that $\eta(X) = 0$ for all $X \in \mathfrak{h}$ and $\eta(\xi) = 1$. From (7) is clear that

$$d_\mathfrak{g}\varphi = d_\mathfrak{h}\omega \wedge \eta + \eta \wedge D.\psi_+ + d_\mathfrak{h}\psi_+. \tag{8}$$

For every triple (e_i, e_j, e_k) of elements of the basis of $\mathfrak{h}$

$$D.\psi_+(e_i, e_j, e_k) = \psi_+(D(e_i), e_j, e_k) + \psi_+(e_i, D(e_j), e_k) + \psi_+(e_i, e_j, D(e_k))$$

where can be checked that if $D \in \mathfrak{sl}(3, \mathbb{C})$ the second member vanishes. Thus, the condition $D \in \mathfrak{sl}(3, \mathbb{C})$ (or equivalently D belongs to the stabilizer Lie algebra $\mathfrak{gl}(\mathfrak{h})_{\psi_+}$ of ψ_+) is considered in order to guarantee that $D.\psi_+ = 0$. Finally in view of (8) we have that

$$d_\mathfrak{g}\varphi = d_\mathfrak{h}\omega \wedge \eta + d_\mathfrak{h}\psi_+,$$

and therefore the G_2 form φ is $d_\mathfrak{g}$ closed if and only if ω and ψ_+ are $d_\mathfrak{h}$ closed, i.e. symplectic half-flat. □

Table 1. Six-dimensional unimodular solvable Lie algebras admitting SHF-structures

Algebra	Structure equations
$\mathfrak{a}$	(0,0,0,0,0,0)
$\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1)$	$(0, 0, -e^{14}, -e^{13}, e^{25}, -e^{26})$
$\mathfrak{g}_{5,1} \oplus \mathbb{R}$	$(0, 0, 0, e^{15}, 0, e^{13})$
$\mathfrak{g}_{5,7}^{-1,-1,1} \oplus \mathbb{R}$	$(-e^{15}, e^{25}, -e^{35}, e^{45}, 0, 0)$
$\mathfrak{g}_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$	$(\alpha e^{15} + e^{35}, -\alpha e^{25} + e^{45}, -e^{15} + \alpha e^{35}, -e^{25} - \alpha e^{45}, 0, 0)$
$\mathfrak{g}_{6,N3}$	$(0, e^{35}, 0, 2e^{15}, 0, e^{13})$
$\mathfrak{g}_{6,38}^{0}$	$(2e^{36}, 0, -e^{26}, -e^{26} + e^{25}, -e^{23} - e^{24}, e^{23})$
$\mathfrak{g}_{6,54}^{0,-1}$	$(e^{16} + e^{45}, -e^{26}, -e^{36} + e^{25}, e^{46}, 0, 0)$
$\mathfrak{g}_{6,118}^{0,-1,-1}$	$(-e^{15} + e^{36}, e^{46} + e^{25}, -e^{16} - e^{35}, e^{45} - e^{26}, 0, 0)$

Remark 3.2. Note that the trace of D, the real representation of certain $A \in \mathfrak{sl}(3, \mathbb{C})$ vanishes. Therefore, the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7$ is unimodular if and only if $\mathfrak{h}$ is so.

In this section, using the previous results we describe new examples of 7-dimensional Lie algebras with closed G_2-structures. These examples are constructed as

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7,$$

where $\mathfrak{h}$ denotes a 6-dimensional solvable Lie algebra with a symplectic half-flat structure and D is a derivation of $\mathfrak{h}$. From [16] the 6-dimensional unimodular solvable Lie algebras with a symplectic half-flat $\mathrm{SU}(3)$-structure are:

Where we use the usual notation in the related literature meaning that if in the k position appears e^{ij} thus $de^k = e^i \wedge e^j$.

The structure equations of the previously mentioned Lie algebras are given in terms of an adapted basis, that is, a basis such that the forms

$$\omega = e^{12} + e^{34} + e^{56},$$
$$\psi_+ = e^{135} - e^{146} - e^{236} - e^{245},$$

are closed and therefore describe a symplectic half-flat $\mathrm{SU}(3)$-structure.

Proposition 3.3. *The Lie algebras described in Table 2 of the Appendix admit the closed* G_2*-structure given by*

$$\varphi = \omega \wedge e^7 + \psi_+.$$

Proof. For every $\mathfrak{h}$, 6-dimensional solvable Lie algebra admitting a symplectic half-flat $\mathrm{SU}(3)$-structure, (see Table 1) we consider Lie algebras

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7,$$

with D being the real representation of certain $A \in \mathfrak{sl}(3, \mathbb{C})$. Thus, the differential operator on $\mathfrak{g}$ can be described as

$$d_{\mathfrak{g}} e^i = d_{\mathfrak{h}} e^i + \sum_{j=1}^{6} D_{ij} e^{j7}, \tag{9}$$

and $\mathfrak{g}$ represents a differential algebra if and only if D is a derivation of $\mathfrak{h}$ or equivalently if the differential operator $d_{\mathfrak{g}}$ vanishes when applied twice. Therefore, in what follows, we present for every Lie algebra in Table 1 the values of the parameters $a_{i,j}$ in D for which $d_{\mathfrak{g}}^2$ vanishes, (equiv. such that the Jacobi identity holds on $\mathfrak{g}$). Finally from Theorem 3.1 the 3-form

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}$$

defines a closed G_2-structure on $\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7$.

- $\mathfrak{h} = \mathfrak{a}$

$$D = \left(\begin{array}{cc|cc|cc} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ -a_{1,2} & a_{1,1} & -a_{1,4} & a_{1,3} & -a_{1,6} & a_{1,5} \\ \hline a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} \\ -a_{3,2} & a_{3,1} & -a_{3,4} & a_{3,3} & -a_{3,6} & a_{3,5} \\ \hline a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & -a_{1,1} - a_{3,3} & -a_{1,2} - a_{3,4} \\ -a_{5,2} & a_{5,1} & -a_{5,4} & a_{5,3} & a_{1,2} + a_{3,4} & -a_{1,1} - a_{3,3} \end{array}\right).$$

$$\begin{aligned} \mathfrak{a} \oplus_D \mathbb{R}e_7 = \big(& a_{1,1}e^{17} + a_{1,2}e^{27} + a_{1,3}e^{37} + a_{1,4}e^{47} + a_{1,5}e^{57} + a_{1,6}e^{67}, \\ & - a_{1,2}e^{17} + a_{1,1}e^{27} - a_{1,4}e^{37} + a_{1,3}e^{47} - a_{1,6}e^{57} + a_{1,5}e^{67}, \\ & a_{3,1}e^{17} + a_{3,2}e^{27} + a_{3,3}e^{37} + a_{3,4}e^{47} + a_{3,5}e^{57} + a_{3,6}e^{67}, \\ & - a_{3,2}e^{17} + a_{3,1}e^{27} - a_{3,4}e^{37} + a_{3,3}e^{47} - a_{3,6}e^{57} + a_{3,5}e^{67}, \\ & a_{5,1}e^{17} + a_{5,2}e^{27} + a_{5,3}e^{37} + a_{5,4}e^{47} + \left(-a_{1,1} - a_{3,3}\right) e^{57} \\ & + \left(-a_{1,2} - a_{3,4}\right) e^{67}, \\ & - a_{5,2}e^{17} + a_{5,1}e^{27} - a_{5,4}e^{37} + a_{5,3}e^{47} + \left(a_{1,2} + a_{3,4}\right) e^{57} \\ & + \left(-a_{1,1} - a_{3,3}\right) e^{67}, 0\big). \end{aligned}$$

- $\mathfrak{h} = \mathfrak{e}(1, 1) \oplus \mathfrak{e}(1, 1)$

$$D = \left(\begin{array}{cc|cc|cc} & & & & & \\ & & & & & \\ \hline & & a_{3,3} & & & \\ & & & a_{3,3} & & \\ \hline & & & & -a_{3,3} & \\ & & & & & -a_{3,3} \end{array}\right).$$

$$\begin{aligned} &\big(\mathfrak{e}(1, 1) \oplus \mathfrak{e}(1, 1)\big) \oplus_D \mathbb{R}e_7 \\ &\quad = (0, 0, -e^{14} + a_{3,3}e^{37}, -e^{13} + a_{3,3}e^{47}, e^{25} - a_{3,3}e^{57}, -e^{26} - a_{3,3}e^{67}, 0). \end{aligned}$$

- $\mathfrak{h} = \mathfrak{g}_{5,1} \oplus \mathbb{R}$

$$D = \left(\begin{array}{cc|cc|cc} & & & & & \\ & & & & & \\ \hline a_{1,3} & & & & a_{3,5} & \\ & a_{1,3} & & & & a_{3,5} \\ \hline a_{1,5} & & a_{3,5} & & & \\ & a_{1,5} & & a_{3,5} & & \end{array}\right).$$

$$\begin{aligned}\left(\mathfrak{g}_{5,1} \oplus \mathbb{R}\right) \oplus_D \mathbb{R}e_7 = (0, 0, a_{1,3}e^{17} + a_{3,5}e^{57}, e^{15} + a_{1,3}e^{27} + a_{3,5}e^{67}, a_{1,5}e^{17} \\ +a_{3,5}e^{37}, e^{13} + a_{1,5}e^{27} + a_{3,5}e^{47}, 0).\end{aligned}$$

- $\mathfrak{h} = \mathfrak{g}_{5,7}^{-1,-1,1} \oplus \mathbb{R}$

$$D = \left(\begin{array}{cc|cc|cc} a_{1,1} & & a_{3,1} & & & \\ & a_{1,1} & & a_{3,1} & & \\ \hline a_{1,3} & & -a_{1,1} & & & \\ & a_{1,3} & & -a_{1,1} & & \\ \hline & & & & & \\ & & & & & \end{array}\right).$$

$$\begin{aligned}\left(\mathfrak{g}_{5,7}^{-1,-1,1} \oplus \mathbb{R}\right) \oplus_D \mathbb{R}e_7 =&(-e^{15} + a_{1,1}e^{17} + a_{3,1}e^{37}, e^{25} + a_{1,1}e^{17} + a_{3,1}e^{37}, \\ &- e^{35} + a_{1,3}e^{17} - a_{1,1}e^{37}, e^{45} + a_{1,3}e^{27} \\ &- a_{1,1}e^{47}, 0, 0, 0).\end{aligned}$$

- $\mathfrak{h} = \mathfrak{g}_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$ **with** $\alpha \geq 0$

$$D = \left(\begin{array}{cc|cc|cc} & & -a_{1,3} & & & \\ & & & -a_{1,3} & & \\ \hline a_{1,3} & & & & & \\ & a_{1,3} & & & & \\ \hline & & & & & \\ & & & & & \end{array}\right),$$

$$\begin{aligned}\left(\mathfrak{g}_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}\right) \oplus_D \mathbb{R}e_7 =&(\alpha e^{15} + e^{35} - a_{1,3}e^{37}, -\alpha e^{25} + e^{45} - a_{1,3}e^{47}, \\ &- e^{15} + \alpha e^{35} + a_{1,3}e^{17}, -e^{25} - \alpha e^{45} \\ &+ a_{1,3}e^{27}, 0, 0, 0),\end{aligned}$$

for all $\alpha > 0$ and

$$D = \left(\begin{array}{cc|cc|cc} & & -a_{1,3} & -a_{1,4} & & \\ & & a_{1,4} & -a_{1,3} & & \\ \hline a_{1,3} & a_{1,4} & & & & \\ -a_{1,4} & a_{1,3} & & & & \\ \hline & & & & & \\ & & & & & \end{array}\right),$$

$$\left(\mathfrak{g}_{5,17}^{0,0,1} \oplus \mathbb{R}\right) \oplus_D \mathbb{R}e_7 = (e^{35} - a_{1,3}e^{37} + a_{1,4}e^{47}, e^{45} - a_{1,4}e^{37} - a_{1,3}e^{47}, - e^{15} + a_{1,3}e^{17} - a_{1,4}e^{27}, -e^{25} + a_{1,4}e^{17} + a_{1,3}e^{27}, 0, 0, 0)$$

for $\alpha = 0$.

- $\mathfrak{h} = \mathfrak{g}_{6,N3}$

$$D = \left(\begin{array}{cc|cc|cc} & & \frac{a_{1,3}}{2} & & -a_{1,5} & \\ & & & \frac{a_{1,3}}{2} & & -a_{1,5} \\ \hline a_{1,3} & & & & 2a_{3,5} & \\ & a_{1,3} & & & & 2a_{3,5} \\ \hline a_{1,5} & & a_{3,5} & & & \\ & a_{1,5} & & a_{3,5} & & \end{array}\right).$$

$$\mathfrak{g}_{6,N3} \oplus_D \mathbb{R}e_7 = \Big(\frac{a_{1,3}}{2}e^{37} - a_{1,5}e^{57}, e^{35} + \frac{a_{1,3}}{2}e^{47} - a_{1,5}e^{67}, a_{1,3}e^{17} + 2a_{3,5}e^{57}, a_{1,3}e^{27} + 2a_{3,5}e^{67}, a_{1,5}e^{17} + a_{3,5}e^{37}, e^{13} + a_{1,5}e^{27} + a_{3,5}e^{47}, 0\Big).$$

- $\mathfrak{h} = \mathfrak{g}_{6,38}^{0}$

$$D = \left(\begin{array}{cc|cc|cc} & & & & & \\ & & & & & \\ \hline & & & & & -a_{3,6} \\ & & & & a_{3,6} & \\ \hline & & & -a_{3,6} & & \\ & & a_{3,6} & & & \end{array}\right).$$

$$\mathfrak{g}_{6,38}^{0} \oplus_D \mathbb{R}e_7 = (2e^{36}, 0, -e^{26} - a_{3,6}e^{67}, -e^{26} + e^{25} + a_{3,6}e^{57}, -e^{23} - e^{24} - a_{3,6}e^{47}, e^{23} + a_{3,6}e^{37}, 0).$$

- $\mathfrak{h} = \mathfrak{g}_{6,54}^{0,-1}$

$$D = \left(0\right).$$

$$\mathfrak{g}_{6,54}^{0,-1} \oplus \mathbb{R}e_7 = (e^{16} + e^{45}, -e^{26}, -e^{36} + e^{25}, e^{46}, 0, 0, 0).$$

- $\mathfrak{h} = \mathfrak{g}_{6,118}^{0,-1,-1}$

$$D = \left(\begin{array}{cc|cc|cc} & & -a_{1,3} & & & \\ & & & -a_{1,3} & & \\ \hline a_{1,3} & & & & & \\ & a_{1,3} & & & & \\ \hline & & & & & \\ & & & & & \end{array}\right).$$

$$\mathfrak{g}_{6,118}^{0,-1,-1} \oplus_D \mathbb{R}e_7 = (-e^{15} + e^{36} - a_{1,3}e^{37}, e^{46} + e^{25} - a_{1,3}e^{47}, -e^{16} - e^{35} + a_{1,3}e^{17}, e^{45} - e^{26} + a_{1,3}e^{27}, 0, 0, 0)$$

□

Remark 3.4. According to [16] there exist 4 Lie algebras and a one-parameter family of solvable non-unimodular Lie algebras admitting symplectic half-flat structures. For all these algebras, with the same procedure described in Proposition 3.3, can be obtained derivations D such that the corresponding 7-dimensional Lie algebra admits a closed G_2-structure. However, these latter algebras are not interesting for our purposes since they will not be unimodular and therefore do not provide compact examples.

3.1. An almost nilpotent compact G_2*-calibrated manifold.*

Let $\mathfrak{h}$ be the 6-dimensional nilpotent Lie algebra defined by the structure equations

$$\mathfrak{h} = (0, e^{35}, 0, 2e^{15}, 0, e^{13}).$$

The almost Hermitian structure (g, J) on $\mathfrak{h}$ given by

$$g = \sum_{i=1}^{6} e^i \otimes e^i, \qquad Je_1 = e_2, \qquad Je_3 = e_4, \qquad Je_5 = e_6 \tag{10}$$

is such that its Kähler form is

$$\omega = e^{12} + e^{34} + e^{56}.$$

Thus, (g, J) together with the complex volume form $\Psi = \psi_+ + i\,\psi_-$, where

$$\psi_+ = e^{135} - e^{146} - e^{236} - e^{245},$$
$$\psi_- = e^{136} + e^{145} + e^{235} - e^{246},$$

define an $\mathrm{SU}(3)$-structure on $\mathfrak{h}$. Clearly, $d\,\omega = d\,\psi_+ = 0$, so $(g, J, \Psi = \psi_+ + i\,\psi_-)$ is a symplectic half-flat $\mathrm{SU}(3)$-structure on $\mathfrak{h}$. Consider now the derivation D of $\mathfrak{h}$ given by

$$\left(\begin{array}{cc|cc|cc} & & 1 & & & \\ & & & 1 & & \\ \hline 2 & & & & & \\ & 2 & & & & \\ \hline & & & & & \\ & & & & & \end{array}\right) \in \mathfrak{sl}(3, \mathbb{C}),$$

that is,

$$D(e_1) = 2e_3, \quad D(e_2) = 2e_4, \quad D(e_3) = e_1, \quad D(e_4) = e_2.$$

Take the Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7,$$

whose structure equations are

$$\mathfrak{g} = (e^{37}, e^{35} + e^{47}, 2e^{17}, 2e^{27} + 2e^{15}, 0, e^{13}, 0).$$

Then, the 3-form φ given by

$$\varphi = \omega \wedge e^7 + \psi_+$$

is a closed G_2 form on $\mathfrak{g}$. Let G be the simply connected solvable Lie group with Lie algebra $\mathfrak{g}$, and let H be the simply connected nilpotent Lie group with Lie algebra $\mathfrak{h}$. Note that $G = \mathbb{R} \ltimes_\phi H$, where ϕ is the unique action $\phi : \mathbb{R} \longrightarrow Aut(H)$ such that, for any $t \in \mathbb{R}$, the morphism $(\phi_t)_*|_e : \mathfrak{h} \longrightarrow \mathfrak{h}$ is given by

$$(\phi_t)_*|_e = \exp(tD),$$

where D is the derivation previously defined on the Lie algebra $\mathfrak{h}$, and exp denotes the map $\exp : \mathrm{Der}(\mathfrak{h}) \to \mathrm{Aut}(\mathfrak{h})$.

In order to show that there exists a discrete subgroup Γ of G such that the quotient space Γ/G is compact we proceed as follows. The SU(3)-basis $\{e_1, \ldots, e_6\}$ of $\mathfrak{h}$ is a rational basis for $\mathfrak{h}$ and, with respect to this basis, we have

$$\phi_t = \left(\begin{array}{cc|cc|cc}
\cosh(\sqrt{2}t) & & \frac{\sqrt{2}}{2}\sinh(\sqrt{2}t) & & & \\
 & \cosh(\sqrt{2}t) & & \frac{\sqrt{2}}{2}\sinh(\sqrt{2}t) & & \\
\hline
\sqrt{2}\sinh(\sqrt{2}t) & & \cosh(\sqrt{2}t) & & & \\
 & \sqrt{2}\sinh(\sqrt{2}t) & & \cosh(\sqrt{2}t) & & \\
\hline
 & & & & 1 & \\
 & & & & & 1
\end{array}\right),$$

To obtain a lattice Γ of G it is enough to find some real number t_0 such that ϕ_t is conjugated to an element $A \in SL(6, \mathbb{Z})$. In these conditions we can find Γ_0 a lattice of H invariant under ϕ_{t_0}, and take

$$\Gamma = (t_0\,\mathbb{Z}) \ltimes_\phi \Gamma_0.$$

In particular, if we consider $t_0 = \frac{\sqrt{2}}{2} \operatorname{arc\cosh}(3)$, then $\cosh(\sqrt{2}\,t_0) = 3$ and $\sinh(\sqrt{2}\,t_0) = 2\sqrt{2}$ and thus ϕ_{t_0} is a matrix whose entries are integer numbers. Therefore, $\mathbb{Z}\langle e_1, \ldots, e_6\rangle$ is a co-compact subgroup of H preserved by ϕ_{t_0}, namely Γ_0. Consequently,

$$\Gamma = (t_0\,\mathbb{Z}) \ltimes_\phi \Gamma_0$$

is a co-compact subgroup of G. Hence, the compact quotient Γ/G is a compact solvmanifold, in particular almost nilpotent. Since $\mathfrak{g}$ is completely solvable

$$H^*_{dR}(\Gamma\backslash G) \cong H^*(\mathfrak{g})$$

and therefore the compact solvmanifold $S = \Gamma\backslash G$ admits a closed G_2-structure.

3.2. A formal almost abelian compact G_2*-calibrated manifold with* $b_1 = 1$.

Let $\mathfrak{h}$ be the 6-dimensional abelian Lie algebra. The almost Hermitian structure (g, J) on $\mathfrak{h}$ given by

$$g = \sum_{i=1}^{6} e^i \otimes e^i, \qquad Je_1 = e_2, \qquad Je_3 = e_4, \qquad Je_5 = e_6$$

is such that its Kähler form is

$$\omega = e^{12} + e^{34} + e^{56}.$$

Thus, (g, J) together with the complex volume form $\Psi = \psi_+ + i\,\psi_-$, where

$$\psi_+ = e^{135} - e^{146} - e^{236} - e^{245},$$
$$\psi_- = e^{136} + e^{145} + e^{235} - e^{246},$$

define an $\mathrm{SU}(3)$-structure on $\mathfrak{h}$. Clearly, $d\,\omega = d\,\psi_+ = 0$, so $(g, J, \Psi = \psi_+ + i\,\psi_-)$ is a symplectic half-flat $\mathrm{SU}(3)$-structure on $\mathfrak{h}$.
Consider now the derivation D of $\mathfrak{h}$ given by

$$\left(\begin{array}{cc|cc|cc} a_{1,1} & & & & & \\ & a_{1,1} & & & & \\ \hline & & a_{3,3} & & & \\ & & & a_{3,3} & & \\ \hline & & & & -a_{1,1}-a_{3,3} & \\ & & & & & -a_{1,1}-a_{3,3} \end{array}\right) \in \mathfrak{sl}(3, \mathbb{C}),$$

that is,

$$D(e_1) = a_{1,1}e_1, \quad D(e_2) = a_{1,1}e_2, \quad D(e_3) = a_{3,3}e_3, \quad D(e_4) = a_{3,3}e_4,$$
$$D(e_5) = (-a_{1,1} - a_{3,3})e_5, \qquad D(e_6) = (-a_{1,1} - a_{3,3})e_6.$$

Take the Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7,$$

whose structure equations are

$$\mathfrak{g} = \Big(a_{1,1}e^{17}, a_{1,1}e^{27}, a_{3,3}e^{37}, a_{3,3}e^{47}, (-a_{1,1} - a_{3,3})e^{57}, (-a_{1,1} - a_{3,3})e^{67}, 0\Big).$$

Then, the 3-form φ given by

$$\varphi = \omega \wedge e^7 + \psi_+$$

is a closed G_2 form on $\mathfrak{g}$. Let us denote by G the simply connected and completely solvable Lie group consisting on matrices of the form

$$a = \left(\begin{array}{cc|cc|cc|c}
e^{a_{1,1}x_7} & & & & & & x_1 \\
 & e^{a_{1,1}x_7} & & & & & x_2 \\
\hline
 & & e^{a_{3,3}x_7} & & & & x_3 \\
 & & & e^{a_{3,3}x_7} & & & x_4 \\
\hline
 & & & & e^{(-a_{1,1}-a_{3,3})x_7} & & x_5 \\
 & & & & & e^{(-a_{1,1}-a_{3,3})x_7} & x_6 \\
\hline
 & & & & & & 1 \; x_7 \\
 & & & & & & 1
\end{array}\right),$$

with $x_i \in \mathbb{R}$, for $i = 1, \ldots, 7$. Then a global system of coordinates $\{x_i\}$ for G is defined by $x_i(a) = x_i$. A standard calculation shows that a basis for the left invariant 1-forms on G can be described by

$$\begin{aligned}
&e^1 = e^{-a_{1,1}x_7}dx_1, \quad e^2 = e^{-a_{1,1}x_7}dx_2, \quad e^3 = e^{-a_{3,3}x_7}dx_3, \quad e^4 = e^{-a_{3,3}x_7}dx_4,\\
&e^5 = e^{(a_{1,1}+a_{3,3})x_7}dx_5, \qquad e^6 = e^{(a_{1,1}+a_{3,3})x_7}dx_6, \qquad and \qquad e^7 = dx_7.
\end{aligned}$$

Therefore $\mathfrak{g}$ is exactly the Lie algebra of G. Notice that $G = \mathbb{R} \ltimes_\phi \mathbb{R}^6$, where $\mathbb{R}$ acts on $\mathbb{R}^6$ via ϕ_t described by

$$\phi_t = \left(\begin{array}{cc|cc|cc}
e^{a_{1,1}t} & & & & & \\
 & e^{a_{1,1}t} & & & & \\
\hline
 & & e^{a_{3,3}t} & & & \\
 & & & e^{a_{3,3}t} & & \\
\hline
 & & & & e^{(-a_{1,1}-a_{3,3})t} & \\
 & & & & & e^{(-a_{1,1}-a_{3,3})t}
\end{array}\right).$$

Thus the operation on the group G is given by

$$\begin{aligned}
r \cdot s = (&s_1 e^{a_{1,1}r_7} + r_1, s_2 e^{a_{1,1}r_7} + r_2, s_3 e^{a_{3,3}r_7} + r_3, s_4 e^{a_{3,3}r_7}\\
&+ r_4, s_5 e^{(-a_{1,1}-a_{3,3})r_7} + r_5, s_6 e^{(-a_{1,1}-a_{3,3})r_7} + r_6, s_7 + r_7),
\end{aligned}$$

where $r = (r_1, \ldots, r_7)$ and $s = (s_1, \ldots, s_7)$.
As in the previous example to obtain a lattice Γ of G it is enough to find some real number t_0 such that ϕ_{t_0} is conjugated to an element $A \in SL(6, \mathbb{Z})$. If Γ_0 denotes a lattice of $\mathbb{R}^6$ invariant under ϕ_{t_0}, take

$$\Gamma = (t_0\,\mathbb{Z}) \ltimes_\phi \Gamma_0.$$

Consider the matrix

$$A = \left(\begin{array}{cc|cc|cc}
1 & & & & 1 & \\
 & 1 & & & & 1 \\
\hline
 & & 2 & & 1 & \\
 & & & 2 & & 1 \\
\hline
1 & & 1 & & 2 & \\
 & 1 & & 1 & & 2
\end{array}\right).$$

Notice that $det(A) = 1$ and its characteristic polinomial is $p(\lambda) = (-\lambda^3 + 5\lambda^2 - 6\lambda + 1)^2$. Take $\tilde{p}(\lambda) = -\lambda^3 + 5\lambda^2 - 6\lambda + 1$, then $\tilde{p}(0) = 1$, $\tilde{p}(1) = -1$, $\tilde{p}(3) = 1$, $\tilde{p}(4) = -7$, thus, from Bolzano's theorem it has three positive real roots. Therefore A has three double positive real eigenvalues λ_1, λ_2 and $\lambda_3 = \frac{1}{\lambda_1\lambda_2}$. Taking appropriated values for t_0 and $a_{3,3}$ ($t_0 = \frac{Ln(\lambda_1)}{a_{1,1}}$, $a_{3,3} = \frac{Ln(\lambda_2)}{Ln(\lambda_1)}a_{1,1}$) we have that $e^{t_0 D} = Diag(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \lambda_3, \lambda_3)$. Since A is symmetric it is diagonalizable and therefore there exist certain P such that $AP = P\phi_{t_0}$. So, the lattice defined by

$$\Gamma_0 = P\,\mathbb{Z}\langle e_1, \ldots, e_6\rangle$$

is invariant under the group $t_0\mathbb{Z}$. Thus

$$\Gamma = (t_0\,\mathbb{Z}) \ltimes_\phi \Gamma_0$$

is a lattice of G. Since $\mathfrak{g}$ is completely solvable Hattori's theorem

$$H^*_{dR}(\Gamma\backslash G) \cong H^*(\mathfrak{g})$$

is satisfied. If we chose the parameter $a_{1,1}$ such that $a_{1,1} \neq 0, -a_{3,3}$ the real cohomology of $S = \Gamma\backslash G$ is exactly

$$\begin{aligned}
&H^0(S) = \langle 1\rangle,\\
&H^1(S) = \langle [e^7]\rangle,\\
&H^2(S) = \langle 1\rangle,\\
&H^3(S) = \langle [e^{135}, e^{136}, e^{145}, e^{146}, e^{235}, e^{236}, e^{245}, e^{246}]\rangle,\\
&H^4(S) = \langle [e^{1357}, e^{1367}, e^{1457}, e^{1467}, e^{2357}, e^{2367}, e^{2457}, e^{2467}]\rangle,\\
&H^5(S) = \langle 1\rangle,\\
&H^6(S) = \langle [e^{123456}]\rangle.
\end{aligned}$$

The corresponding minimal model of S is the graded algebra $(\mathcal{M}, d)$, with $\mathcal{M}$ the free algebra

$$\mathcal{M} = \bigwedge(a) \otimes \bigwedge(c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8) \otimes \bigwedge V^{\geq 5}$$

with a of degree 1 and c_i have degree 3. The morphism

$$\rho : \mathcal{M} \longrightarrow \Omega(S)$$

that induces an isomorphism in cohomology is be defined by

$$\begin{array}{lll}
\rho(a) = e^7, & \rho(c_1) = e^{135}, & \rho(c_2) = e^{136},\\
\rho(c_3) = e^{145}, & \rho(c_4) = e^{146}, & \rho(c_5) = e^{235},\\
\rho(c_6) = e^{236}, & \rho(c_7) = e^{245}, & \rho(c_8) = e^{246}.
\end{array}$$

Recalling Definition 2.6

$$\begin{array}{ll}
C^1 = a, & N^1 = 0,\\
C^2 = 0, & N^2 = 0,\\
C^3 = c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, & N^3 = 0,
\end{array}$$

thus S is 3-formal and by Theorem 2.7 it is formal. Then, the compact solvmanifold S has first Betti number 1, is formal and admits a closed G_2-structure.

Proposition 3.5. *The compact solvmanifold S does not admit any invariant torsion-free* G_2*-structure.*

Proof. We prove that S has no cocalibrated G_2-structures and therefore does not admit torsion-free G_2-structures either. In [1, Lemma 3.3] it is proved the following restriction to the existence of a cocalibrated G_2-structure on a Lie algebra $\mathfrak{l}$: if there exists a pair of different non zero vectors $X, Y \in \mathfrak{l}$ such that

$$(\iota_Y(\iota_X \gamma))^2 = 0, \tag{11}$$

for every closed 4-form γ where ι denotes the contraction operator, then the Lie algebra $\mathfrak{l}$ does not admit cocalibrated G_2-structures. This restriction is obvious from the fact that if a cocalibrated G_2-structure exists, there is a closed 4-form such that in terms of an adapted basis can be described canonically. Thus for every pair of different non zero vectors, Eq. (11) cannot vanish. On the other hand, from the structure equations of $\mathfrak{g}$, the Lie algebra associated to G with $S = \Gamma\backslash G$, can be checked that the space of closed 4-forms is

$$\begin{aligned} Z^4(\mathfrak{g}^*) = \langle & e^{1237}, e^{1247}, e^{1257}, e^{1267}, e^{1347}, e^{1357}, e^{1367}, e^{1457}, e^{1467}, e^{1567}, \\ & e^{2347}, e^{2357}, e^{2367}, e^{2457}, e^{2467}, e^{2567}, e^{3457}, e^{3467}, e^{3567}, e^{4567}\rangle. \end{aligned}$$

Let $\gamma \in Z^4(\mathfrak{g}^*)$ then

$$(\iota_{e_1}(\iota_{e_2}\gamma))^2 = 0,$$

and thus $\mathfrak{g}$ has no cocalibrated G_2-structures and, in particular, it cannot admit torsion-free G_2-structures. Therefore the solvmanifold $S = \Gamma\backslash G$ has no invariant torsion-free G_2-structures. □

4. Lie algebras with a cocalibrated G_2-structure

In this section we show that if a 6-dimensional half-flat Lie algebra is endowed with a particular type of derivation, then a Lie algebra with a coclosed G_2-structure can be constructed.

We recall that a coclosed G_2-structure on a real Lie algebra $\mathfrak{g}$ of dimension 7 consists on the presence of a G_2 form which is coclosed. In order to obtain an expression adapted to our purposes, in this section we characterize a G_2 form on $\mathfrak{g}$ as a 3-form that can be written as

$$\varphi = e^{127} + e^{347} + e^{567} + e^{246} - e^{235} - e^{145} - e^{136},$$

with respect to some basis $\{e^1, \ldots, e^7\}$ of the dual space of $\mathfrak{g}$.

Let us also recall that $\mathfrak{sp}(6, \mathbb{R})$ is given by

$$\mathfrak{sp}(6,\mathbb{R}) = \left\{ \left(\begin{array}{cc|cc|cc} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ a_{2,1} & -a_{1,1} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} \\ \hline -a_{2,4} & a_{1,4} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} \\ a_{2,3} & -a_{1,3} & a_{4,3} & -a_{3,3} & a_{4,5} & a_{4,6} \\ \hline -a_{2,6} & a_{1,6} & -a_{4,6} & a_{3,6} & a_{5,5} & a_{5,6} \\ a_{2,5} & -a_{1,5} & a_{4,5} & -a_{3,5} & a_{6,5} & -a_{5,5} \end{array} \right), \text{ with } a_{i,j} \in \mathbb{R} \right\}. \tag{12}$$

Theorem 4.1. *Let $\mathfrak{h}$ be a 6-dimensional Lie algebra and let $\mathfrak{g}$ be a 7-dimensional Lie algebra satisfying*

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7,$$

with D a derivation of $\mathfrak{h}$ such that $D \in \mathfrak{sp}(6, \mathbb{R})$. Then the following two conditions are equivalent:

(1) *The* SU(3)*-structure on $\mathfrak{h}$ given by*

$$\omega = e^{12} + e^{34} + e^{56},$$
$$\psi_+ = e^{135} - e^{146} - e^{236} - e^{245},$$

is half-flat.

(2) *The* G_2*-structure on $\mathfrak{g}$ given by*

$$\varphi = \omega \wedge e^7 - \psi_-,$$

is coclosed.

Proof. If we identify k-forms on $\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}\xi$ which annihilate ξ with k-forms on $\mathfrak{h}$, one may write any k-form $\gamma \in \Lambda^k \mathfrak{g}^*$ as

$$\gamma = \alpha \wedge \xi^\flat + \beta$$

for unique $\alpha \in \Lambda^{k-1}\mathfrak{h}^*$ and $\beta \in \Lambda^k \mathfrak{h}^*$ where $\flat$ denotes the canonical isomorphism. One can check that

$$d_{\mathfrak{g}}\gamma = d_{\mathfrak{h}}\alpha \wedge \xi^\flat + \xi^\flat \wedge D.\beta + d_{\mathfrak{h}}\beta$$

for $D.\beta$ being the natural action of $D \in \mathfrak{gl}(\mathfrak{h})$ on $\beta \in \Lambda^k \mathfrak{h}^*$.

Thus, consider the SU(3)-structure (ω, ψ_+) on $\mathfrak{h}$ such that with respect to the basis $\{e_1, \ldots, e_6\}$ has the canonical expression. Consider also, the G_2 form

$$\varphi = \omega \wedge \eta - \psi_-,$$

with η the 1-form such that $\eta(X) = 0$ for all $X \in \mathfrak{h}$ and $\eta(\xi) = 1$. Thus we have that

$$*\varphi = \frac{1}{2}\omega^2 + \psi_+ \wedge \eta$$

and for (7) is clear that

$$d_{\mathfrak{g}}(*\varphi) = d_{\mathfrak{h}}\Big(\frac{\omega^2}{2}\Big) + \eta \wedge D.\Big(\frac{\omega^2}{2}\Big) + d_{\mathfrak{h}}\psi_+ \wedge \eta. \tag{13}$$

For every cuadruplet (e_i, e_j, e_k, e_l) of elements of the basis of $\mathfrak{h}$

$$\begin{aligned} D.\omega^2(e_i, e_j, e_k, e_l) = {} & \omega^2(D(e_i), e_j, e_k, e_l) + \omega^2(e_i, D(e_j), e_k, e_l) \\ & + \omega(e_i, e_j, D(e_k), e_l) + \omega^2(e_i, e_j, e_k, D(e_l)), \end{aligned}$$

where can be checked that if $D \in \mathfrak{sp}(6, \mathbb{R})$ the second member vanishes. Thus, the condition $D \in \mathfrak{sp}(6, \mathbb{R})$ (or equivalently D belongs to the stabilizer Lie algebra $\mathfrak{gl}(\mathfrak{h})_{\frac{\omega^2}{2}} = \mathfrak{gl}(\mathfrak{h})_{\omega}$ of ω) is considered in order to guarantee that $D.\omega^2 = 0$. Finally in view of (13) we have that

$$d_{\mathfrak{g}}(*\varphi) = d_{\mathfrak{h}}\left(\frac{\omega^2}{2}\right) + d_{\mathfrak{h}}\psi_+ \wedge \eta,$$

and therefore the G_2 form φ is $d_{\mathfrak{g}}$ coclosed if and only if ω^2 and ψ_+ are $d_{\mathfrak{h}}$ closed, i.e. half-flat. □

Previous theorem describes a method to construct 7-dimensional Lie algebras with a coclosed G_2-structure.

Remark 4.2. Note that the trace of $D \in \mathfrak{sp}(6, \mathbb{R})$ vanishes. Therefore, the Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7$ will be unimodular if and only if $\mathfrak{h}$ is so.

4.1. An almost abelian compact G_2*-cocalibrated manifold.*

Let $\mathfrak{h}$ be the 6-dimensional abelian Lie algebra. The almost Hermitian structure given in (10) defines an $SU(3)$-structure on $\mathfrak{h}$. Concretely, since ω^2 and ψ_+ are closed it is a half-flat structure. Consider now the derivation D of $\mathfrak{h}$ given by

$$diag(1, -1, 1, -1, 1, -1)$$

that is,

$$\begin{array}{lll} D(e_1) = e_1, & D(e_2) = -e_2, & D(e_3) = e_3, \\ D(e_4) = -e_4, & D(e_5) = e_5 \quad and & D(e_6) = -e_6. \end{array}$$

Thus, the Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7,$$

which is described by the structure equations

$$\mathfrak{g} = (e^{17}, -e^{27}, e^{37}, -e^{47}, e^{57}, -e^{67}, 0),$$

is completely solvable and from Theorem 4.1 admits the coclosed G_2 form

$$\varphi = \omega \wedge \eta - \psi_-,$$

This coclosed G_2 form was already obtained in [20] where the author gave a complete classification of coclosed G_2-structures on Lie algebras with a codimensional one Abelian ideal. In what follows we describe explicitly the corresponding compact solvmanifold admitting such structure.

Let us denote by G the simply connected and completely solvable Lie group consisting on matrices of the form

$$a = \left(\begin{array}{cc|cc|cc|cc} e^{x_7} & & & & & & & x_1 \\ & e^{-x_7} & & & & & & x_2 \\ \hline & & e^{x_7} & & & & & x_3 \\ & & & e^{-x_7} & & & & x_4 \\ \hline & & & & e^{x_7} & & & x_5 \\ & & & & & e^{-x_7} & & x_6 \\ \hline & & & & & & 1 & x_7 \\ & & & & & & & 1 \end{array}\right),$$

with $x_i \in \mathbb{R}$, for $i = 1, \dots, 7$. Then a global system of coordinates $\{x_i\}$ for G is defined by $x_i(a) = x_i$. A standard calculation shows that a basis for the left invariant 1-forms on G can be described by

$$\begin{aligned} &e^1 = e^{-x_7} dx_1, \quad e^2 = e^{x_7} dx_2, \quad e^3 = e^{-x_7} dx_3, \quad e^4 = e^{x_7} dx_4, \\ &e^5 = e^{-x_7} dx_5, \quad e^6 = e^{x_7} dx_6, \qquad and \qquad e^7 = dx_7. \end{aligned}$$

Therefore $\mathfrak{g}$ is exactly the Lie algebra of G. Notice that $G = \mathbb{R} \ltimes_\phi \mathbb{R}^6$, where $\mathbb{R}$ acts on $\mathbb{R}^6$ via ϕ_t described by

$$diag(e^t, e^{-t}, e^t, e^{-t}, e^t, e^{-t}).$$

Thus the operation on the group G is given by

$$\begin{aligned} a \cdot b = (&b_1 e^{a_7} + a_1, b_2 e^{-a_7} + a_2, b_3 e^{a_7} + a_3, b_4 e^{-a_7} \\ &+ a_4, b_5 e^{a_7} + a_5, b_6 e^{-a_7} + a_6, b_7 + a_7), \end{aligned}$$

where $a = (a_1, \dots, a_7)$ and $b = (b_1, \dots, b_7)$.
To construct a lattice Γ of G it is enough to find some real number t_0 such that ϕ_{t_0} is conjugated to an element $A \in SL(6, \mathbb{Z})$. If Γ_0 denotes a lattice of $\mathbb{R}^6$ invariant under ϕ_{t_0}, take

$$\Gamma = (t_0 \mathbb{Z}) \ltimes_\phi \Gamma_0.$$

Consider the matrix

$$A = \left(\begin{array}{cc|cc|cc} 2 & 1 & & & & \\ 1 & 1 & & & & \\ \hline & & 2 & 1 & & \\ & & 1 & 1 & & \\ \hline & & & & 2 & 1 \\ & & & & 1 & 1 \end{array}\right),$$

with triple eigenvalues $\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}$. Taking $t_0 = Ln(\frac{3+\sqrt{5}}{2})$ we have that ϕ_{t_0} and A are conjugated. In particular, take

$$P = \left(\begin{array}{cc|cc|cc} 1 & \frac{-1+\sqrt{5}}{2} & & & & \\ 1 & \frac{-1-\sqrt{5}}{2} & & & & \\ \hline & & 1 & \frac{-1+\sqrt{5}}{2} & & \\ & & 1 & \frac{-1-\sqrt{5}}{2} & & \\ \hline & & & & 1 & \frac{-1+\sqrt{5}}{2} \\ & & & & 1 & \frac{-1-\sqrt{5}}{2} \end{array}\right),$$

it is easy to check that $PA = \phi_{t_0} P$. So, the lattice defined by

$$\Gamma_0 = P\,\mathbb{Z}\langle e_1, \ldots, e_6\rangle$$

is invariant under the group $t_0\mathbb{Z}$. Thus

$$\Gamma = (t_0\,\mathbb{Z}) \ltimes_\phi \Gamma_0$$

is a lattice of G. Since $\mathfrak{g}$ is completely solvable

$$H^*_{dR}(\Gamma\backslash G) \cong H^*(\mathfrak{g})$$

and the compact solvmanifold $S = \Gamma\backslash G$ admits a coclosed G_2-structure.

4.2. An almost nilpotent compact G_2*-cocalibrated manifold.*

Let $\mathfrak{h}$ be the 6-dimensional nilpotent Lie algebra defined by the structure equations

$$\mathfrak{h} = (0, e^{35}, 0, 2e^{15}, 0, e^{13}).$$

The almost Hermitian structure (g, J) described in (10)

$$\begin{aligned} \omega &= e^{12} + e^{34} + e^{56}, \\ \psi_+ &= e^{135} - e^{146} - e^{236} - e^{245}, \\ \psi_- &= e^{136} + e^{145} + e^{235} - e^{246}, \end{aligned}$$

is a symplectic half-flat $\mathrm{SU}(3)$-structure on $\mathfrak{h}$. Consider now the derivation D of $\mathfrak{h}$ given by

$$\left(\begin{array}{cc|cc|cc} & & & & 2 & \\ & & & & & -1 \\ \hline & & & & & \\ & & & & & \\ \hline 1 & & & & & \\ & -2 & & & & \end{array}\right) \in \mathfrak{sp}(6, \mathbb{R}),$$

that is,

$$D(e_1) = e_5, \quad D(e_2) = -2e_6, \quad D(e_5) = 2e_1, \quad D(e_6) = -e_2.$$

Take the Lie algebra

$$\mathfrak{g} = \mathfrak{h} \oplus_D \mathbb{R}e_7,$$

whose structure equations are

$$\mathfrak{g} = (2e^{57}, e^{35} - e^{67}, 0, 2e^{15}, e^{17}, e^{13} - 2e^{27}, 0).$$

Then, the 3-form φ given by

$$\varphi = \omega \wedge \eta - \psi_-,$$

is a coclosed G_2 form on $\mathfrak{g}$. Let G be the simply connected solvable Lie group with Lie algebra $\mathfrak{g}$, and let H be the simply connected nilpotent Lie group with Lie algebra $\mathfrak{h}$. Note that $G = \mathbb{R} \ltimes_\phi H$, with

$$\phi_t = \left(\begin{array}{cc|cc|cc} \cosh(\sqrt{2}t) & & & & \frac{\sqrt{2}}{2}\sinh(\sqrt{2}t) & \\ & \cosh(\sqrt{2}t) & & & & -\sqrt{2}\sinh(\sqrt{2}t) \\ \hline & & 1 & & & \\ & & & 1 & & \\ \hline \sqrt{2}\sinh(\sqrt{2}t) & & & & \cosh(\sqrt{2}t) & \\ & -\frac{\sqrt{2}}{2}\sinh(\sqrt{2}t) & & & & \cosh(\sqrt{2}t) \end{array}\right),$$

in particular, if we consider $t_0 = \frac{\sqrt{2}}{2} \operatorname{arc\,cosh}(3)$, then $\cosh(\sqrt{2}\, t_0) = 3$ and $\sinh(\sqrt{2}\, t_0) = 2\sqrt{2}$ and thus ϕ_{t_0} is a matrix whose entries are integer numbers. Therefore, $\mathbb{Z}\langle e_1, \ldots, e_6\rangle$ is a co-compact subgroup of H preserved by ϕ_{t_0}, namely Γ_0. Consequently,

$$\Gamma = (t_0\, \mathbb{Z}) \ltimes_\phi \Gamma_0$$

is a co-compact subgroup of G. Hence, the compact quotient $\Gamma\backslash G$ is a compact solvmanifold, in particular almost nilpotent. Since $\mathfrak{g}$ is completely solvable

$$H^*_{dR}(\Gamma\backslash G) \cong H^*(\mathfrak{g})$$

and therefore the compact solvmanifold $S = \Gamma\backslash G$ admits a coclosed G_2-structure.

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Appendix

see Table 2.

Table 2. Lie algebras endowed with a closed G_2-structure obtained in Proposition 3.3, not coming from the 6-dimensional abelian Lie algebra

$\mathfrak{g}$	Structure equations
$\mathfrak{e}(1,1) \oplus \mathfrak{e}(1,1) \oplus_D \mathbb{R}e_7$	$(0, 0, -e^{14} + a_{3,3}e^{37}, -e^{13} + a_{3,3}e^{47}, e^{25} - a_{3,3}e^{57}, -e^{26} - a_{3,3}e^{67})$
$(\mathfrak{g}_{5,1} \oplus \mathbb{R}) \oplus_D \mathbb{R}e_7$	$(0, 0, a_{1,3}e^{17} + a_{3,5}e^{57}, e^{15} + a_{1,3}e^{27} + a_{3,5}e^{67}, a_{1,5}e^{17} + a_{3,5}e^{37}, e^{13} + a_{1,5}e^{27} + a_{3,5}e^{47}, 0)$
$(\mathfrak{g}_{5,7}^{-1,-1,1} \oplus \mathbb{R}) \oplus_D \mathbb{R}e_7$	$(-e^{15} + a_{1,3}e^{17} + a_{1,1}e^{37}, e^{25} + a_{1,1}e^{17} + a_{3,1}e^{37}, -e^{35} + a_{1,3}e^{17} + a_{1,1}e^{37}, e^{45} + a_{1,3}e^{27} - a_{1,1}e^{47}, 0, 0, 0)$
$\left(\mathfrak{g}_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}\right) \oplus_D \mathbb{R}e_7$ $\forall \alpha > 0$	$(\alpha e^{15} + e^{35} - a_{1,3}e^{37}, -\alpha e^{25} + e^{45} - a_{1,3}e^{47}, -e^{15} + \alpha e^{35} + a_{1,3}e^{17}, -e^{25} - \alpha e^{45} + a_{1,3}e^{27}, 0, 0, 0)$
$\left(\mathfrak{g}_{5,17}^{0,0,1} \oplus \mathbb{R}\right) \oplus_D \mathbb{R}e_7$	$(e^{35} - a_{1,3}e^{37} + a_{1,4}e^{47}, e^{45} - a_{1,4}e^{37} - a_{1,3}e^{47}, -e^{15} + a_{1,3}e^{17} - a_{1,4}e^{27}, -e^{25} + a_{1,4}e^{17} + a_{1,3}e^{27}, 0, 0, 0)$
$\mathfrak{g}_{6,N3} \oplus_D \mathbb{R}e_7$	$(\frac{a_{1,3}}{2}e^{37} - a_{1,5}e^{57}, e^{35} + \frac{a_{1,3}}{2}e^{47} - a_{1,5}e^{67}, a_{1,3}e^{17} + 2a_{3,5}e^{57}, a_{1,3}e^{27} + 2a_{3,5}e^{67}, a_{1,5}e^{17} + a_{3,5}e^{37}, e^{13} + a_{1,5}e^{27} + a_{3,5}e^{47}, 0)$
$\mathfrak{g}_{6,38}^{0} \oplus \mathbb{R}e_7$	$(2e^{36}, 0, -e^{26} - a_{3,6}e^{67}, -e^{26} + e^{25} + a_{3,6}e^{57}, -e^{23} - e^{24} - a_{3,6}e^{47}, e^{23} + a_{3,6}e^{37}, 0)$
$\mathfrak{g}_{6,54}^{0,-1} \oplus \mathbb{R}e_7$	$(e^{16} + e^{45}, -e^{26}, -e^{36} + e^{25}, e^{46}, 0, 0, 0)$
$\mathfrak{g}_{6,118}^{0,-1,-1} \oplus_D \mathbb{R}e_7$	$(-e^{15} + e^{36} - a_{1,3}e^{37}, e^{46} + e^{25} - a_{1,3}e^{47}, -e^{16} - e^{35} + a_{1,3}e^{17}, e^{45} - e^{26} + a_{1,3}e^{27}, 0, 0, 0)$

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